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A Lagrangian relaxation approach to a coupled lot-sizing and cutting stock problem

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ABSTRACT

Industrial production processes involving both lot-sizing and cutting stock problems are common in many industrial settings. However, they are usually treated in a separate way, which could lead to costly production plans. In this paper, a coupled mathematical model is formulated and a heuristic method based on Lagrangian relaxation is proposed. Computational results prove its effectiveness.

1. Introduction

This work focuses on decision-making problems associated with the production planning at tactical/operational levels. In this context, consider a production process in which the main activity is to manufacture final products assembled from parts which are to be cut from large objects in stock. More particularly, in furniture industries this cutting process consists of cutting wooden rectangular plates into smaller ordered parts, as shown in Fig. 1.

Based on the known demand for final products per period, a lot-sizing problem (LSP) should be solved to decide the quantity of each final product which has to be manufactured in each period of the planning horizon. The objective is to minimize the production, inventory, and setup costs. However, it should be noted that the LSP does not optimize the material lost in the cutting process, since different lots require different amounts of parts that lead to a diverse material loss.

In this paper we propose an integrated methodology to optimize the LSP and the imbedded cutting stock problem (CSP) simultaneously. The solution of this combined problem explores the trade-off when the CSP is solved by considering the trim loss, as well as the production, inventory and setup costs of final products for several periods. For example, if we consider the manufacturing anticipation of some final products, the storage costs increase, but probably the trim loss reduces due to better cutting patterns and setup costs are also expected to be lowered. This problem is called coupled lot-sizing and cutting stock problem or, for short, lot-cut problem (LCP).

Typically, industries solve this problem separately by first solving the LSP, determining the production planning of final products for each period, and then solving the CSP, making decisions (for each period) on how to cut each plate in order to meet the quantity of parts necessary to fulfill the final product demand. However, when solving the problem in a decomposed fashion, possible infeasibilities can arise related to saw machine capacity, i.e., in some periods the planned saw capacity can be greater than the given capacity. These possible infeasibilities due to violated capacity constraints are overcome by transferring production among periods (this approach is reviewed

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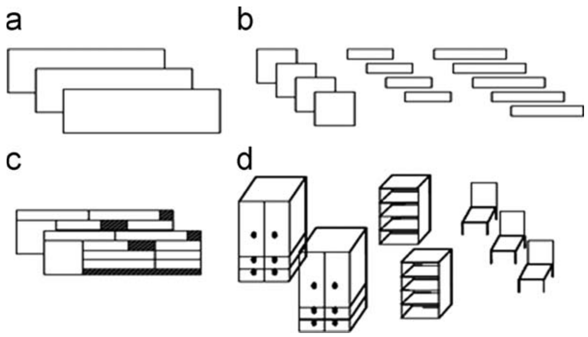

Fig. 1. (a) Plates in stock; (b) ordered parts; (c) cutting patterns; and (d) final products.

in Section 3). Alternatively, difficulties in obtaining feasible solutions can be overcome with overtime work (considering one more work shift) or backlogging.

This alternative can be modeled by considering new variables to increase the machine capacities, or by tackling with negative stocks, but this is not considered in this paper.

The integration of the cutting stock and the production planning processes has not yet been much discussed in the literature, but its relevance, found in different industrial settings suggests that it is a very interesting and important problem to be researched. This type of problem is usually found in furniture industries, when wooden parts should be cut to assemble the final products (e.g., industrial/residential furniture), in fiber glass industries that cut fiber glass plates to manufacture printed circuit boards, in aluminum window frame manufacturing where aluminum profiles are cut to make several window types, in the packaging industry where carton plates are cut in order to fulfill a carton box demand, and so forth. In all mentioned problems, the lot-sizing and CSPs are economically relevant in the process.

Drexl and Kimms (1997) suggest many coupled problems (*coordination problems*) for future research, for example the CSP integrated into the LSP, describing them as "probably the most crucial objective for future work". Some work involving cutting stock decisions in production planning have been found in the literature (see Arbib and Marinelli, 2005; Hendry et al., 1996; Nonas and Thorstenson, 2000, 2008; Poltroniere et al., 2008; Reinders, 1992) but, either they do not consider capacity constraints, or they consider the cutting patterns known as a priori, or they use a two-phase solution procedure. None of them solve both problems in conjunction considering capacity constraints, setup, storage, and trim loss costs.

Gramani and França (2006) analyzed the trade-off that arises when solving the CSP by taking into account the production planning for various periods. The goal was to minimize the trim loss costs in the cutting process, the inventory costs (for parts) and the setup costs. The authors formulated a mathematical model of the combined cutting stock and LSP and proposed a solution method based on an analogy with the network shortest path problem, comparing its results with the ones simulated in the industrial practice. However, this paper does not consider the final products.

Analogously, other problems at a tactical/operational level can be linked with a view to a better global solution. Recently, Toledo et al. (2008) have presented an optimization model for the integrated lot-sizing and scheduling problem in a soft drink industry. The challenge of this work was to determine simultaneously the minimum lot-sizing and scheduling costs of raw material in tanks and also in the bottling lines, where setup costs and setup times are sequence-dependent.

Pileggi et al. (2005) also studied another combined problem. The authors presented three heuristic approaches to deal with the integrated cutting pattern generation and sequencing problem, taking into consideration the trade-off between trim loss and the number of open stacks. Although the investigation of combined problems is very relevant, the studies found in the literature are still not vast.

This paper is organized as follows: in the next section, a mixed-integer mathematical model for the LCP is proposed. Then, in Section 3 a decomposition heuristic (DH) is presented, which reflects the industrial practice for solving the lot-sizing and CSPs separately. In Section 4, a heuristic based on Lagrangian relaxation is described, which is able to find good quality solutions in quite reasonable computational times. Finally, computational comparisons using four sets of randomly generated instances are presented.

2. Mathematical modeling

In this section, we present a new approach for the combined cutting stock and LSP. Consider a stock of rectangular plates of length L and width W (Fig. 1a) to be cut into m rectangular ordered parts of lengths l_p and widths w_p, $p = 1,2,\ldots,P$ (Fig. 1b). The order for parts to be cut in each period depends on the decision of how many final products are manufactured per period. The CSP consists of cutting plates into smaller parts so that the ordered parts are met and a certain function is optimized (e.g., the trim loss, the cost of plates cut). The way a plate is cut provides a cutting pattern (Fig. 1c), and how many plates are cut according to a cutting pattern is a decision variable. Our model of the combined problem does not use a previous set of cutting patterns generated *a priori*.

Let T be the number of periods, M the number of different types of final products demanded, P the number of different types of parts, and N the number of all possible cutting patterns. Considering an index variation as $t = 1,\ldots,T$, $i = 1,\ldots,M$, $p = 1,\ldots,P$, $j = 1,\ldots,N$, the problem parameters and variables are defined as:

Parameters:
d_{it}: demand for final product i in period t;
r_{pi}: number of parts of type p necessary to compose a unit of final product i;
b_t: saw machine capacity expressed as the total amount of material area possible to be cut in period t;
a_{pj}: number of parts of type p in cutting pattern j;

cp: unit cost of the plate to be cut;
c_{it}: unit production cost of final product i in period t;
h_{it}: unit inventory cost of final product i in period t;
s_{it}: setup cost of final product i in period t;
$L \times W$: length and width of the plate;
$l_p \times w_p$: length and width of the part of type p.

Variables:
x_{it}: number of final products i to be manufactured in period t;
I_{it}: number of final products i stocked at the end of period t;
y_{jt}: number of plates cut according to pattern j in period t;
z_{it}: binary variable: $z_{it} = 1$ if $x_{it} > 0$; zero, otherwise.

The following constraints should be considered.

Inventory balance of final products:

$$x_{it} + I_{i,t-1} - I_{it} = d_{it}, \quad i = 1, \ldots, M; \ t = 1, \ldots, T \tag{1}$$

These constraints assure that the demand of final product i in period t (d_{it}) is met without delay, i.e., $I_{it} \geqslant 0$, $i = 1,\ldots,M$; $t = 1,\ldots,T$. Without loss of generality, the initial inventory can be considered zero. Note that in this paper, the storage costs for the parts are not considered. Having in mind that the most important parcel of the storage cost is the alternate use of the immobilized capital incorporated in final products, the storage cost for the parts is implicitly considered in the final product holding cost.

Parts demand:

$$\sum_{j=1}^{N} a_{pj} y_{jt} \geqslant \sum_{i=1}^{M} r_{pi} x_{it}, \quad p = 1, \ldots, P; \ t = 1, \ldots, T \tag{2}$$

The left hand side is the number of type p parts cut which has to be greater than or equal to the number of required type p parts on the right hand side.

Saw machine capacity:

$$\sum_{i=1}^{M} v_i x_{it} \leqslant b_t, \quad t = 1, \ldots, T \tag{3}$$

where $v_i = \sum_{p=1}^{P} (r_{pi} l_p w_p), \quad i = 1, \ldots, M$ is the material area needed to make a unit of final product i. These constraints assure the material area cut is less or equal to the saw machine capacity (b_t).

Setup and production linkage:

$$x_{it} \leqslant Q z_{it}, \quad i = 1, \ldots, M; \ t = 1, \ldots, T \tag{4}$$

where Q is a large number.

These constraints assure that $z_{it} = 1$, if $x_{it} > 0$. In case of $x_{it} = 0$, the condition $z_{it} = 0$ follows to reach optimality. Note that in this paper the aim of the cutting process is to minimize only the trim loss so the setup in this process is not considered. Observe that if the cutting process also aims to minimize the number of pattern changes then the setup in the cutting process becomes more relevant.

Objective:

$$\sum_{i=1}^{M} \sum_{t=1}^{T} (c_{it} x_{it} + h_{it} I_{it} + s_{it} z_{it}) + \sum_{j=1}^{N} \sum_{t=1}^{T} cpLW y_{jt} \tag{5}$$

The objective function to be minimized is the overall cost of production, inventory, and setup and the costs derived from the plate's use.

The mixed-integer mathematical model (LCP) can be written then as follows:

$$Z = \text{Min} \quad \sum_{i=1}^{M} \sum_{t=1}^{T} (c_{it} x_{it} + h_{it} I_{it} + s_{it} z_{it}) + \sum_{j=1}^{N} \sum_{t=1}^{T} cpLW y_{jt} \tag{6}$$

$$\text{s.t.} \quad x_{it} + I_{i,t-1} - I_{it} = d_{it}, \quad i = 1, \ldots, M; \ t = 1, \ldots, T \tag{7}$$

$$\sum_{j=1}^{N} a_{pj} y_{jt} \geqslant \sum_{i=1}^{M} r_{pi} x_{it}, \quad p = 1, \ldots, P; \ t = 1, \ldots, T \tag{8}$$

$$\sum_{i=1}^{M} v_i x_{it} \leqslant b_t, \quad t = 1, \ldots, T \tag{9}$$

$$x_{it} \leqslant Q z_{it}, \quad i = 1, \ldots, M; \ t = 1, \ldots, T \tag{10}$$

$$x_{it}, I_{it} \geqslant 0; \quad z_{it} \in \{0, 1\}, \quad i = 1, \ldots, M; \ t = 1, \ldots, T \tag{11}$$

$$y_{jt} \geqslant 0, \quad j = 1, \ldots, N; \ t = 1, \ldots, T \tag{12}$$

The integer condition on decision variables x_{it}, I_{it}, and y_{jt} can be relaxed if demands are high, but two difficulties still remain: the enormous quantity of cutting patterns (a_{pj}) that could be generated and the presence of 0–1 setup variables. Note that constraints (8) are those that couple decisions of lot-sizing and cutting. Next, we give two approaches to heuristically solve problem (6)–(12).

3. Solution methods

3.1. Decomposition heuristic (DH): LSP and CSP solved separately

Typically practical decision makers build their production plans in two sequential steps:

- *Step* 1: Solve the LSP: first part of (6) subject to: (7), (9)–(11) and obtain x_{it};
- *Step* 2: With x_{it} fixed in Step 1, solve the CSP: second part of (6) subject to (8), (12).

Although solving the problems in a separated form is simpler due to the existence of efficient algorithms in the literature, it can increase the global costs, especially if the plate's costs are relevant in the total costs of the product. In the furniture industry, for example, the plate's cost can correspond to approximately 50% of the final product cost.

3.2. Lagrangian heuristic (LH)

Let $\mu_{pt} \geqslant 0$, $p = 1,\ldots,P$; $t = 1,\ldots,T$ and $\lambda_t \geqslant 0$, $t = 1,\ldots,T$ be the Lagrangian multipliers associated with constraints (8) and (9), respectively. Then, the Lagrangian problem, denoted by $L(\boldsymbol{\lambda},\boldsymbol{\mu})$, can be written as follows:

$$Z_{L(\lambda,\mu)} = \text{Min} \quad \sum_{i=1}^{M}\sum_{T=1}^{T}(c_{it}x_{it} + h_{it}I_{it} + s_{it}z_{it}) + \sum_{j=1}^{N}\sum_{t=1}^{T} cpLWy_{jt} + \sum_{t=1}^{T}\lambda_t\left(\sum_{i=1}^{M} v_i x_{it} - b_t\right) + \sum_{p=1}^{P}\sum_{t=1}^{T}\mu_{pt}\left(\sum_{i=1}^{M} r_{pi}x_{it} - \sum_{j=1}^{N} a_{pj}y_{jt}\right) \tag{13}$$

or

$$\text{Min} \quad \sum_{i=1}^{M}\sum_{T=1}^{T}\left[\left(c_{it} + \lambda_t v_i + \sum_{p=1}^{P}\mu_{pt}r_{pi}\right)x_{it} + h_{it}I_{it} + s_{it}z_{it}\right] + \sum_{j=1}^{N}\sum_{t=1}^{T}\left(cpLW - \sum_{p=1}^{P}\mu_{pt}a_{pj}\right)y_{jt} - \sum_{t=1}^{T}\lambda_t b_t \tag{14}$$

$$\text{s.t.} \quad x_{it} + I_{i,t-1} - I_{it} = d_{it}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{15}$$

$$x_{it} \leqslant Qz_{it}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{16}$$

$$x_{it}, I_{it} \geqslant 0; \quad z_{it} \in \{0,1\}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{17}$$

$$y_{jt} \geqslant 0, \quad j = 1,\ldots,N; \; t = 1,\ldots,T \tag{18}$$

The Lagrangian problem can be decomposed into the following two sub-problems:

1. *Lot-sizing sub-problem* LS $(\boldsymbol{\lambda},\boldsymbol{\mu})$:

$$Z_{LS(\lambda,\mu)} = \text{Min} \quad \sum_{i=1}^{M}\sum_{t=1}^{T}\left[\left(c_{it} + \lambda_t v_i + \sum_{p=1}^{P}\mu_{pt}r_{pi}\right)x_{it} + h_{it}I_{it} + s_{it}z_{it}\right] - \sum_{t=1}^{T}\lambda_t b_t \tag{19}$$

$$\text{s.t.} \quad x_{it} + I_{i,t-1} - I_{it} = d_{it}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{20}$$

$$x_{it} \leqslant Qz_{it}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{21}$$

$$x_{it}, I_{it} \geqslant 0; \quad z_{it} \in \{0,1\}, \quad i = 1,\ldots,M; \; t = 1,\ldots,T \tag{22}$$

Note that LS$(\boldsymbol{\lambda},\boldsymbol{\mu})$ can be decomposed into M independent sub-problems, for each product i, and each one can be satisfactory solved by dynamic programming (Evans, 1985; Wagner and Whitin, 1958).

2. *Cutting sub*-problem $\mathbf{C}(\boldsymbol{\mu})$:

$$Z_{C(\mu)} = \text{Min} \quad \sum_{j=1}^{N}\sum_{t=1}^{T}\left(cpLW - \sum_{p=1}^{P}\mu_{pt}a_{pj}\right)y_{jt} \tag{23}$$

$$\text{s.t.} \quad y_{jt} \geqslant 0, \quad j = 1,\ldots,N; \; t = 1,\ldots,T \tag{24}$$

Therefore, $Z_L(\lambda,\mu) = Z_{LS(\lambda,\mu)} + Z_{C(\mu)}$, which is a lower bound to LCP (6)–(12). The problem of determining the maximum lower bound is the dual Lagrangian problem, to be solved by the sub-gradient method.

Note that the cutting sub-problem $\mathbf{C}(\boldsymbol{\mu})$ has an obvious lower bound which can be obtained as follows: if $(cpLW - \sum_{p=1}^{P}\mu_{pt}a_{pj}) < 0$, then $y_{jt} \to \infty$ and $Z_{C(\mu)} \to -\infty$ and, consequently, $Z_{L(\lambda,\mu)} \to -\infty$. Therefore, μ_{pt} should be chosen in such a way that $(cpLW - \sum_{p=1}^{P}\mu_{pt}a_{pj}) \geqslant 0$.

The condition $(cpLW - \sum_{p=1}^{P}\mu_{pt}a_{pj}) \geqslant 0$ is the optimality condition to the tth CSP, denoted by $CS(x^t)$, $t = 1,\ldots,T$:

$$Z_{CS(x)} = \text{Min} \quad \sum_{j=1}^{N}\sum_{t=1}^{T} cpLWy_{jt} \tag{25}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} a_{pj}y_{jt} \geqslant \sum_{i=1}^{M} r_{pi}x_{it}, \quad p = 1,\ldots,P \tag{26}$$

$$y_{j,t} \geqslant 0, \quad j = 1,\ldots,N \tag{27}$$

where $\mathbf{x}^t = (x_{it})_{i=1,\ldots,M}$ is the solution of LS$(\boldsymbol{\lambda},\boldsymbol{\mu})$. Therefore, CS$(\mathbf{x}^t)$, $t = 1,\ldots,T$ are solved instead of $C(\boldsymbol{\mu})$ and their dual variables, $\boldsymbol{\mu}^t = (\mu_{pt})_{p=1,\ldots,P}$ are used as the Lagrangian multipliers to $L(\boldsymbol{\lambda},\boldsymbol{\mu})$. Note that with this choice it follows that: $Z_{C(\mu)} = 0$. Consequently, $Z_{L(\lambda,\mu)} = Z_{LS(\lambda,\mu)}$.

The solution approach consists of solving the Lagrangian problem $Z_L(\lambda,\mu)$ for the initial iteration given λ_t and μ_{pt}, first obtaining a solution. If this solution satisfies the relaxed constraints and if the values of the objective functions of the original problem and the Lagrangian problem are the same (within a tolerance) so the optimal solution for the original problem is achieved. Otherwise, the Lagrangian multipliers must be updated.

In order to update λ dual variables and maximize the lower bound $Z_{L(\lambda,\mu)}$ the sub-gradient method is used (observe that the λ variables are associated to constraints (9), and the term $\sum_{t=1}^{T}\lambda_t(\sum_{i=1}^{M}v_i x_{it} - b_t)$ is added to the Lagrangian objective function).

The search direction is given by the projection of the sub-gradient onto non-negative variables: $\boldsymbol{\phi} = (\phi_t)_{t=1,\ldots,T}$, where $\phi_t = \max\{0, \sum_{i=1}^{M}v_i x_{it} - b_t\}$. Therefore

$$\lambda_t \leftarrow \lambda_t + \varepsilon\phi_t, \quad t = 1,\ldots,T \tag{28}$$

where the step $\varepsilon = \pi(Z_{UB} - Z_{LB})/\sum_{t=1}^{T}\phi_t^2$ and initially $\pi = 2$, and halved after 10 iterations without improvements. As a monotone improvement of the dual function is not guaranteed, the best solution found should be saved.

The stopping criteria used for the sub-gradient method according to Camerini et al. (1975) are:

1. If the step size is less than a tolerance ($\varepsilon < 10^{-5}$).
2. If the maximum iteration number is achieved (maximum of 100 iterations).
3. If the lower and upper bounds differ from a tolerance (Z_{UB}-$Z_{LB} < 10^{-5}$).

It is worth remarking that when LS$(\boldsymbol{\lambda},\boldsymbol{\mu})$ is solved obtaining Z_{LB}, probably an unfeasible solution is found by violating saw machine capacities (9). Therefore, a smoothing heuristic to recover feasibility is used. This heuristic consists of two steps, regressive and progressive, to transfer lots to previous or later periods, if possible, according to capacity availability (see (Trigeiro et al., 1989), for transferences based on Wagner–Whitin optimality properties or (Araujo and Arenales, 2000) for transferences based on slack complementary optimality properties). This heuristic provides a feasible solution and so an upper bound Z_{UB}.

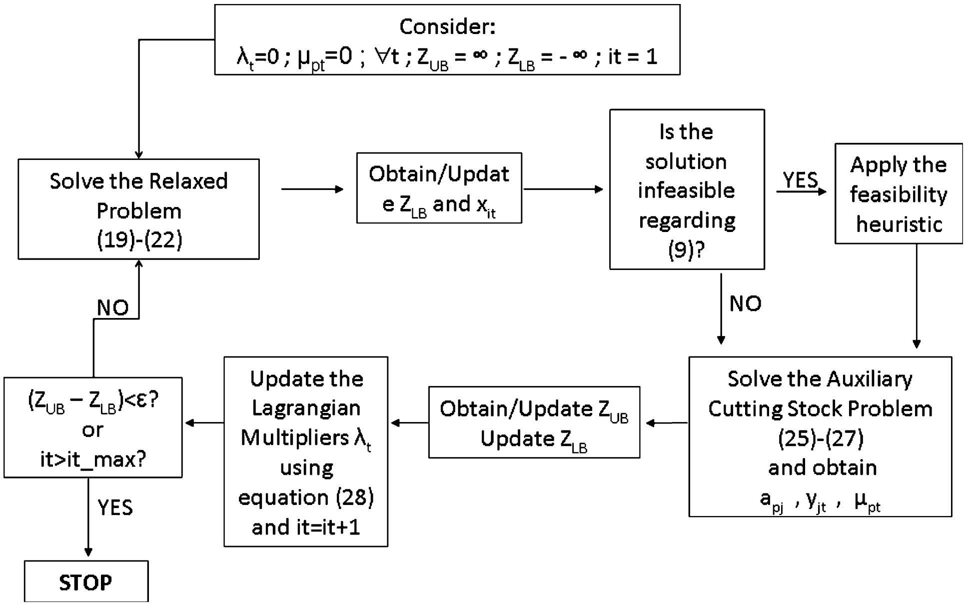

Fig. 2. Schematic representation of the Lagrangian-based heuristic to LCP.

4. A Lagrangian-based heuristic to LCP

The previous developments give us a heuristic to solve LCP (6)–(12), which is summarized as follows:

1. Choose initial multipliers λ_t and μ_{pt}. The iteration number index is represented by *it*.
2. Solve LS($\boldsymbol{\lambda}$,$\boldsymbol{\mu}$), and obtain $\mathbf{x}^t = (x_{it})_{i=1,\ldots,M}$, $t = 1,\ldots,T$. Apply a smoothing heuristic to obtain a feasible solution.
3. Solve CS($\mathbf{x}^t$), $t = 1,\ldots,T$, and obtain $\boldsymbol{\mu}^t = (\mu_{pt})_{p=1,\ldots,P}$.
4. Update λ according to the sub-gradient method.
5. If a stopping criterion is met then stop, otherwise go to step 2.

The flow chart in Fig. 2 represents the Lagrangian-based heuristic.

It should be noted that Z_{LB} and Z_{UB} values do not give monotonic sequences along the iterations of the LH. To confirm that the properties of lower and upper bounds for Z_{LB} and Z_{UB} are guaranteed for each iteration, their values are updated only when they are strictly better than the current values.

5. Computational results

The proposed heuristics were implemented using the programming language C and commercial package CPLEX 7.5 was used for solving the CSP.

Computational tests were executed in four sets with 10 instances each. The first and the second sets (small instances) consider 7 periods for the planning horizon, 20 different types of parts and 10 final products, where the width (w_p) and the length (l_p) of the parts are randomly generated in the interval [25,50] for the first set and in [10,75] for the second test set. The third and fourth sets (large instances) consider 7 periods, 30 different types of parts and 15 final products, where the width (w_p) and the length (l_p) of parts are generated in the interval [25,50] for the third set and in [10,75] for the fourth set.

The problem parameters were randomly generated using the following intervals:

- r_{pi} in interval [0,5].
- d_{it} in interval [0,200] and d_{i1} in interval [0,50] (in order to reduce infeasibilities right in the first period).
- The final product cost (c_{it}) is given by the area of raw material that product i requires: $c_{it} = \sum_{p=1}^{P} r_{pi} l_p w_p$ for $i = 1,\ldots,M$; $t = 1,\ldots,T$. Of course, any price per unit area could be used. Note that, for simplification, c_{it} is independent of t.
- The final product inventory costs were defined as: $h_{it} = 0.001 c_{it}$, for $i = 1,\ldots,M$; $t = 1,\ldots,T$.
- The plate length and width were defined as 100.
- The plate cost was taken as: $cp = LW/10$.
- The setup cost is the same for all the final products and periods, and is given by $s_{it} = 10^{*} c_{11}$.
- The cutting capacity b_t was generated as the mean value of the capacity used, when the lot-for-lot method is applied to determine the lot sizes. It produces exactly the respective demand in each period, implicating zero storage in all periods. It is assumed $b_1 = b_1 + 0.3 b_1$, in order to reduce infeasibilities in the first period.
- In all examples the maximum number of allowed iterations in the sub-gradient method, equal to 100, is utilized.

The CSPs were restricted to 2-stage guillotine cutting patterns. A cut is of guillotine type if when applied to a rectangle, two new rectangles are produced (Figs. 3a and b). A guillotine-cutting pattern is obtained from sequential guillotine cuts (Fig. 3c).

Guillotine cutting patterns are called 2-stage if just one change in the direction of guillotine cuts is allowed (see Fig. 4).

A 2-stage guillotine-cutting pattern gives the coefficients a_{pj} in constraints (8). The model (6)–(12) allows for any other kind of cutting patterns, but in this case, the machine capacity is affected by the number of stages (cut orientations) and number of cuts (see Morabito and Arenales, 2000 for practical cutting patterns in furniture industries). The decomposition method of Gilmore and Gomory (1965) can be used to determine a 2-stage guillotine cutting pattern. For an alternative methodology see Morabito and Arenales (1995).

The validation of the lot-cut approach proposed in this paper is made through the comparison of the results

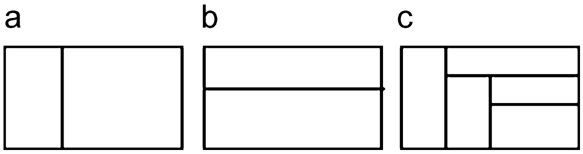

Fig. 3. (a) Vertical guillotine cut; (b) horizontal guillotine cut; and (c) guillotine cutting pattern.

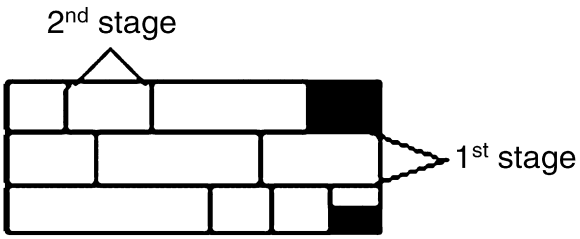

Fig. 4. A 2-stage guillotine-cutting pattern.

obtained with the LH with those obtained using the DH, that is, a comparison between the new combined approach and the methodology normally used by industry. It should be noted that the feasible solution obtained in the first iteration of LH corresponds to the DH solution. For a broader visibility of the comparisons, the DH results without the feasibility step were also shown. This will be known as *DH solution without feasibility*.

Tables 1–4 show the results for the three heuristics—DH without feasibility, DH and LH—when executing the four instance sets, where:

- The column #*setups* denotes the number of setups.
- The column #*plates* presents the number of cut plates.
- The column *storage cost* denotes the total inventory cost.
- The columns *setup variation* and *storage variation* present the relative deviation obtained by the LH solution when compared with the DH solution.

As for simplification c_{it} is assumed to be independent of t, the production cost is not included in the results.

Tables 1 and 2 represent the results obtained with instance set 1 and 2 (small) and Tables 3 and 4 with instance sets 3 and 4 (large). In the captions of each table, the mean time for execution of one instance is indicated. Firstly observe the relevance of the feasibility heuristic (Trigeiro et al., 1989) denoted by the differences between the values obtained in columns #*setup* and *storage costs* for the *DH solution without feasibility* and *DH solution*, respectively. In order to get feasible solutions the number of setups has to be increased considerably resulting in substantial reductions in storage costs.

When comparing the DH solution with the LH solution, it can be observed that the main reductions are obtained in the number of setups, while the storage costs do not show such an impressive improvement and the number of plates remains almost the same. Tables 1 and 2 show that

Table 1
20 pieces, 10 final products, 7 periods, $25 \leqslant l,\ w \leqslant 50$.

	DH solution without feasibility		DH solution			LH solution			Setup variation (%)	Storage variation (%)
	#Setup	Storage costs	#Setup	#Plates	Storage costs	#Setup	#Plates	Storage costs		
1	15	85,298	29	9,490	37,486	27	9,490	37,405	7.41	0.22
2	13	155,753	28	10,313	68,469	24	10,313	68,964	16.67	−0.72
3	14	153,838	31	12,156	64,647	27	12,156	59,321	14.81	8.98
4	12	153,149	27	9,423	49,930	27	9,423	48,310	0.00	3.35
5	12	164,244	37	9,998	46,834	28	9,998	44,953	32.14	4.18
6	13	171,600	29	12,183	64,953	26	12,183	66,264	11.54	−1.98
7	14	123,364	27	9,739	53,213	26	9,739	52,351	3.85	1.65
8	15	107,842	36	10,335	64,818	25	10,335	64,704	44.00	0.18
9	14	143,296	35	12,181	56,321	27	12,181	56,251	29.63	0.12
10	12	135,959	29	7,852	54,985	24	7,852	47,837	20.83	14.94
Mean value	13	139,435	31	10,367	56,165	26	10,367	54,636	18.09	3.09

Mean time of 4 s per instance.

Table 2
20 pieces, 10 final products, 7 periods, $10\leqslant l,\ w\leqslant 75$.

	DH solution without feasibility		DH solution			LH solution			Setup variation (%)	Storage variation (%)
	#Setup	Storage costs	#Setup	#Plates	Storage costs	#Setup	#Plates	Storage costs		
1	15	112,718	30	15,560	46,780	28	15,631	47,786	7.14	−2.11
2	11	327,759	26	18,920	117,144	24	18,910	114,377	8.33	2.42
3	13	225,234	29	21,098	85,791	25	20,995	81,850	16.00	4.82
4	11	226,223	28	14,264	80,197	24	14,559	78,716	16.67	1.88
5	12	237,530	32	18,216	65,528	27	18,302	54,212	18.52	20.87
6	12	180,848	28	13,137	53,790	26	13,122	53,681	7.69	0.20
7	15	119,731	29	11,996	59,232	26	11,964	56,333	11.54	5.15
8	12	160,713	29	12,245	55,354	27	12,234	57,000	7.41	−2.89
9	15	115,945	44	12,885	71,195	25	12,762	66,867	76.00	6.47
10	11	230,777	35	18,341	44,242	30	18,319	43,656	16.67	1.34
Mean value	13	193,748	31	15,676	67,925	26	15,680	65,448	18.60	3.82

Mean time of 25 s per instance.

Table 3
30 pieces, 15 final products, 7 periods, $25\leqslant l,\ w\leqslant 50$.

	DH solution without feasibility		DH solution			LH solution			Setup variation (%)	Storage variation (%)
	#Setup	Storage costs	#Setup	#Plates	Storage costs	#Setup	#Plates	Storage costs		
1	18	332,221	40	22,459	98,752	39	22,459	99,575	2.56	−0.83
2	16	456,497	44	24,778	107,813	37	24,778	109,875	18.92	−1.88
3	15	349,031	47	17,504	63,739	38	17,504	65,419	23.68	−2.57
4	21	260,703	35	20,338	135,317	34	20,338	139,320	2.94	−2.87
5	16	451,588	46	23,976	115,057	40	23,976	112,934	15.00	1.88
6	17	361,365	55	21,800	88,806	40	21,800	88,289	37.50	0.59
7	18	362,117	42	22,438	96,719	38	22,438	98,601	10.53	−1.91
8	19	296,087	43	21,140	107,740	35	21,140	110,384	22.86	−2.40
9	21	321,903	55	26,130	104,039	39	26,130	104,467	41.03	−0.41
10	16	391,568	47	22,502	82,211	39	22,502	83,692	20.51	−1.77
Mean value	18	358,308	45	22,307	100,019	38	22,307	101,256	19.55	−1.22

Mean time of 5 s per instance.

the relative deviation with respect to the setups is about 18%, i.e., the LH has obtained a reduction of 18% in the number of setups in comparison with the DH solution. Storage costs were reduced by a modest 2–3%.

In Tables 3 and 4, where a greater number of parts is considered—and thus having more possibilities for their arrangement in the plates, it can also be noticed that the Lagrangian solution increases the inventory costs, which is compensated with a reduction in the number of setups (by achieving better cutting patterns).

A quality measure of the solutions obtained by the proposed heuristic LH and the DH approach is the relative deviation between the lower and upper bounds accomplished by them:

$$\mathrm{GAP} = \frac{Z_{UB} - Z_{LB}}{Z_{LB}} * 100\% \qquad (29)$$

It should be observed that even when a very good solution is obtained, it is possible to have a solution of low quality due to a poor upper bound.

Fig. 5 presents the mean deviation between the lower and upper bounds attained by DH and LH. Obviously, the GAP LH is always smaller than or equal to the GAP DH, but in most cases LH has achieved a significant improvement.

It can be noticed also that in some cases the GAP LH is zero or very close to zero. This means that the proposed LH could get solutions that are very close to the optimum of the combined problem.

6. Conclusions

In this paper a new mathematical model for the combined lot-sizing and cutting stock problem is presented. The model incorporates the conjecture that it is advantageous to anticipate the production of certain final product lots, which on the one hand will increase the inventory costs, but on the other hand will compensate with the reduction of the setup and cutting costs (using a better arrangement in cutting patterns and trim losses).

Table 4
30 pieces, 15 final products, 7 periods, $10 \leqslant l, w \leqslant 75$.

	DH solution without feasibility		DH Solution			LH Solution			Setup variation (%)	Setup variation (%)
	#Setup	Storage costs	#Setup	#Plates	Storage costs	#Setup	#Plates	Storage costs		
1	15	585,135	42	33,353	121,270	39	33,355	121,731	7.69	−0.38
2	15	623,905	52	37,560	125,523	38	37,670	127,632	36.84	−1.65
3	16	508,470	38	29,624	148,955	36	29,420	149,158	5.56	−0.14
4	18	577,871	40	39,336	231,326	33	39,379	238,516	21.21	−3.01
5	16	475,641	46	30,703	112,882	39	30,642	112,504	17.95	0.34
6	17	448,470	44	30,348	108,707	38	30,213	107,910	15.79	0.74
7	16	604,976	50	39,728	124,447	40	39,358	128,858	25.00	−3.42
8	17	336,231	41	24,627	87,242	36	24,516	90,336	13.89	−3.42
9	19	557,003	40	42,306	165,131	38	42,297	161,401	5.26	2.31
10	19	458,315	41	37,812	164,999	38	37,929	163,238	7.89	1.08
Mean value	17	517,602	43	34,540	139,048	38	34,478	140,128	15.71	−0.76

Mean time of 32 s per instance.

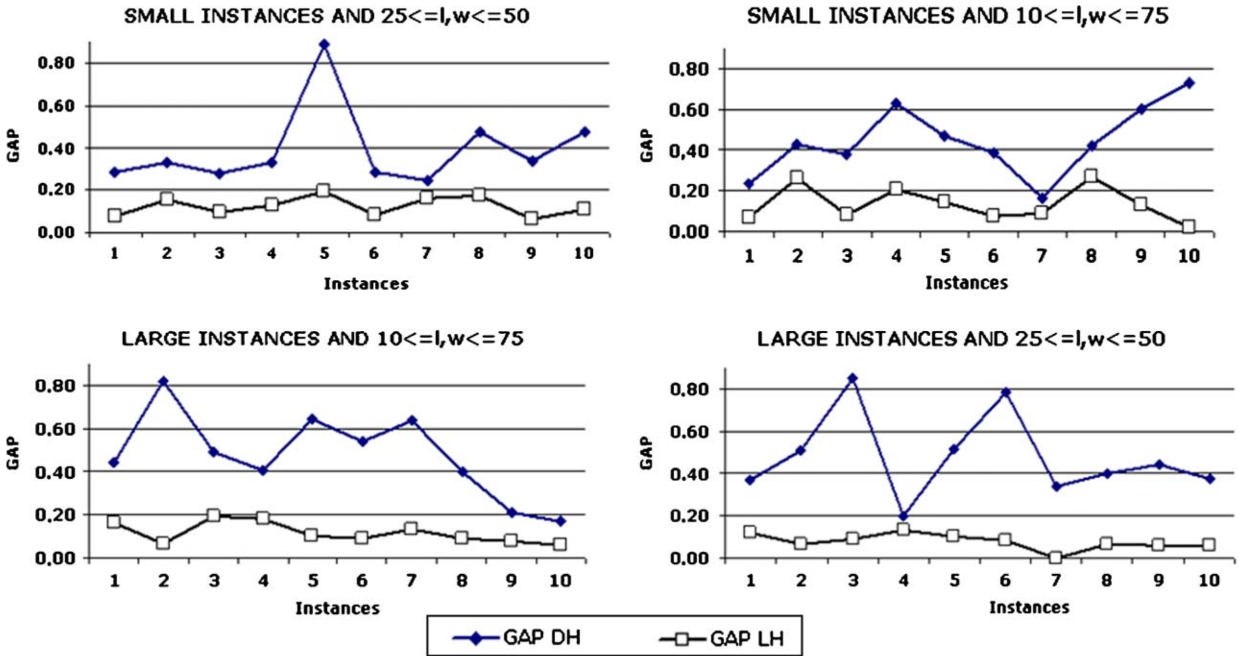

Fig. 5. Comparisons between lower and upper bounds.

For this model, a resolution method based on Lagrangian relaxation is proposed, using the sub-gradient method to update the Lagrange multiplier vectors. In order to compare the results of the proposed Lagrangian heuristic (LH), the decomposition heuristic (DH) was used. The DH heuristic reflects the industrial practice of treating the problem separately, first solving the lot-sizing problem and then, for each period of the planning horizon, solving a cutting stock problem. That solution is provided by the first iteration of the Lagrangian heuristic.

The effectiveness of the proposal can be shown by the computational results obtained for 4 test sets by two different analyses: (1) the relative deviation of the setup, plates and storage quantities obtained by the DH and LH solutions and (2) the gain obtained by the LH solution when compared with the DH solution. The findings for large problems reveal that the approach that deals with the problems in a combined form (LH) presents a small increase in the storage costs, which is more than compensated by a substantial reduction in the setup costs and a small decrease in the number of cut plates. For smaller problems (represented by Tables 1 and 2) the results were similar, in this case, even for the storage cost a reduction was observed (around 3%). It is worth noticing that for some instances, the LH was able to find optimal solutions.

In short, in many manufacturing processes where cutting decisions have a great impact on production costs (such as furniture industries, leather product industries process, and packaging industries), storage, setup costs, and trim losses are relevant decisions that managers wish to achieve in the minimum cost plan. To make this decision more helpful, we show in this paper that instead of solving these problems in a separate way, the Lagrangian heuristic aggregates most of the relevant decisions, and leads to a lower total cost plan.

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